39 p.

SMITHSONIAN INSTITUTION
ASTROPHYSICAL OBSERVATORY

Research in Space Science

SPECIAL REPORT

Number 144

GEODESICS ON AN EQUIPOTENTIAL SURFACE OF REVOLUTION

by

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February 10, 1964

CAMBRIDGE, MASSACHUSETTS 02138

SAO Special Report No. 144

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Walter Köhnlein[2]

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Abstract.--An equipotential surface of revolution is propagated as a reference surface for a worldwide geodetic system. A trigonometry--using geodesics as surface curves--is briefly outlined, and the geodetic applications are given in the most important cases. The developments are valid for any length of the geodesics customarily used for "long lines" on an ellipsoid of revolution.

The figure of the Earth is usually identified with the geometrical structure of a certain equipotential surface of the gravity field, called the geoid. In geodesy this geoid is used as a reference surface for geodetic measurements, but in numerical computations, an easier manipulating surface such as a sphere or an ellipsoid of revolution, is usually substituted for the geoid. Hence these latter reference surfaces must fulfill two major conditions:

1) They must represent the mean structure of the geoid in an area under consideration as well as possible, and

2) They must allow a simple computation procedure when geodesics are used as surface curves.

In this paper we investigate a reference surface that not only fulfills the above conditions but also shows some characteristics that make it especially suitable for worldwide geodetic systems. Author

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[1]This work was supported in part by grant NsG 87-60 from the National Aeronautics and Space Administration.

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A. The Equipotential Surface of Revolution

We assume for the moment that our surface of revolution is zonewise monotonous and steadily curved. Then the behavior of a geodesic is described by Clairaut's equation

$$p \sin A = \text{const.}, \qquad (1)$$

which means that in each point of the surface the product of the parallel radius p and the sine of the azimuth A of a geodesic is constant and equal along its whole run. Let us specialize the radius of the revolution surface to be the curve locus in which the mean of the geopotential--around each latitude circle--becomes identical with the potential of the geoid. Then the obtained equipotential surface of revolution represents the mean structure of the whole geoid, and the radii toward the north and south poles become equal to the corresponding radii of the geoid if we put the coordinate origin into the gravity center of the Earth.

Approximating the geopotential function of the gravity field of the Earth in the following form:

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\bar{N}} \sum_{m=0}^{k} \left(\frac{a}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\sin \beta)\right\} + \frac{\omega^2 r^2}{2} \cos^2\beta \qquad (2)$$

with

a = equatorial (maximum parallel) radius

μ = product of the mass of the Earth with the gravitational constant

ω = angular velocity of the Earth

r = geocentric radius, i.e., the distance from the gravity center to a point in the free space

β = geocentric latitude, i.e., the complement of the angle that is included by r and the rotation axis of the Earth

λ = longitude, i.e., the angle between a meridian plane through r and an arbitrarily fixed meridian plane (Greenwich)

C_{nm}, S_{nm} = harmonic coefficients

P_{nm} = Legendre's associated function and

$$P_{nm}(\sin\beta) = \cos^m\beta \frac{d^m P_n(\sin\beta)}{d(\sin\beta)^m} \tag{3}$$

when

$$P_n(\sin\beta) = \frac{1}{2^n n!} \sum_{t=0}^{k} \frac{(2n-2t)!}{(n-2t)!} \binom{n}{t} \left(-1\right)^t \sin^{n-2t}\beta \tag{4}$$

with

$$k = \frac{n}{2} \ (n = \text{even})$$

$$k = \frac{n-1}{2} \ (n = \text{odd}),$$

then we obtain the mean potential along a latitude circle

$$\bar{U} = \frac{1}{2\pi} \int_{\lambda=0}^{2\pi} \left\{ \frac{\mu}{r} \left[1 + \sum_{n=2}^{\bar{N}} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\sin\beta) \right] + \frac{\omega^2 r^2}{2} \cos^2\beta \right\} d\lambda \ , \tag{5}$$

or by putting

$$\bar{U} = V = \text{const.}$$

$$P_{no} = P_n \ \text{(Legendre's coefficient)}$$

$$V = \frac{\mu}{r} \left\{ 1 + \sum_{n=2}^{\bar{N}} \left(\frac{a}{r}\right)^n C_{no} P_n(\sin\beta) \right\} + \frac{\omega^2 r^2}{2} \cos^2\beta \ . \tag{6}$$

This last equation already includes our reference surface in question. To pick it out among the other equipotential surfaces we have to give one of the following:

1) its geopotential V; or
2) the gravity either in the equator or in a certain latitude; or
3) the (parallel) radius in the equator or in a certain latitude.

The values of the gravity or the (parallel) radius are understood here as mean values along a certain latitude circle according to our previous definition of the equipotential surface of revolution.

For practical reasons the first point must be excluded because the potential of the geoid cannot be directly measured on its surface. But the second and third possibility can be well realized either by pendulum measurements or by distance and angle measurements along the equator or a latitude circle for example.

In the next section we treat point 3 first to the extent that handling of points 2 and 3 becomes equal for further developments. In this stage we outline briefly the procedure when the gravity is given.

B. Equation of the Meridian Curve

A revolution surface is completely described by its meridian curve. Because equation (6) is continuous in r within the interval $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$ we can always uniformly approximate the meridian curve--in the same interval--by polynomials (Weierstrass's approximation theorem).

1. Equation of the meridian curve in polar coordinates, with the origin in the gravity center.

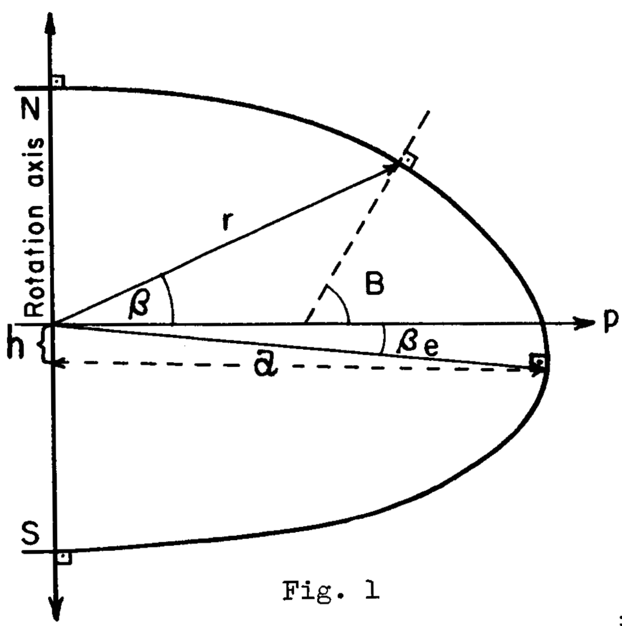

Fig. 1

We construct the meridian curve in the explicit form:

$$r = r(\beta) \qquad (7)$$

and develop (7) formally into a polynomial of degree $\overline{M}$:*

$$r = a\left(1 + \sum_{i=0}^{\overline{M}} r_i \sin^i\beta\right) . \qquad (8)$$

Introducing (8) together with (4) into (6) in the slightly changed form:

$$r = \frac{\mu}{V}\left\{1 + \frac{\omega^2 r^3}{2\mu}(1 - \sin^2\beta) + \sum_{n=2}^{\overline{N}} \left(\frac{a}{r}\right)^n C_{no}\, P_n(\sin\beta)\right\} , \qquad (9)$$

we obtain the unknown coefficients r_i by comparison of the terms having the same power of sin β. This can be simplified by making use of a series

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*$\overline{M}$ depends on the intended numerical accuracy.

development

$$r^j = a^j \sum_{k=0}^{\infty} \binom{j}{k} \left(\sum_{i=0}^{\overline{M}} r_i \sin^i \beta \right)^k , \tag{10}$$

which is convergent for

$$- 1 \leq \sin \beta \leq + 1$$

and any value of j. In general we obtain

$$r_i = r_i(r_0, r_1, \ldots r_{i-1}; V) \tag{11}$$

or, for example, the first few terms for $\overline{N} = 4$, $\overline{M} = 8$ and an accuracy of about 7 digits:

$$\left.\begin{aligned}
&\left(r_0 = \frac{\mu}{Va}\left(1 + \frac{\omega^2 a^3}{2\mu} - \frac{C_{20}}{2} + \frac{3}{8} C_{40}\right) - 1 + \ldots\right) \leq 10^{-10} \\
&r_1 = - \frac{3}{2} C_{20} \frac{\mu}{Va} \Big/ \left[1 - \frac{\mu}{Va}\left(\frac{3\omega^2 a^3}{2\mu} + C_{20}\right)\right] + \ldots \\
&r_2 = \left(\frac{3}{2} C_{20} - \frac{\omega^2 a^3}{2\mu} - \frac{15}{4} C_{40}\right) \frac{\mu}{Va} \Big/ \left[1 - \frac{\mu}{Va}\left(\frac{3\omega^2 a^3}{2\mu} + C_{20} - \frac{3}{2} C_{40}\right)\right] + \ldots \\
&r_3 = \ldots. \\
&\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{12}$$

We may mention that in (11) the potential V is still unknown. A method of computation is given in section 3 .

We now discuss the assumptions made in chapter A about the monotony and curvature of the surface of revolution. From equation (8) follows that, in geodetic application, the meridian curve is monotonous between

$$0 \leq \beta \leq \frac{\pi}{2}$$

and

$$- \frac{\pi}{2} \leq \beta \leq 0$$

while the curvature is steady over the whole interval

$$- \frac{\pi}{2} \leq \beta \leq \frac{\pi}{2} .$$

The last result derives from the convergence and steadiness of the second derivatives of (8), taken along the meridian curve. As a result the differential equations of the geodesic in chapter C can be integrated on the whole surface without any restrictions.

2. The geographic latitude B as a function of the geocentric latitude β, and vice versa.

The relation in question

$$\sin B = \sin B\ (\sin \beta) \tag{13}$$

can be easily derived from figure 1:

$$\sin B = \frac{r \sin \beta - \frac{dr}{d\beta} \cos \beta}{\sqrt{r^2 + \left(\frac{dr}{d\beta}\right)^2}}\ , \tag{14}$$

which leads, together with

$$\frac{dr}{d\beta} = a \cos \beta \sum_{i=1}^{\overline{M}} i\ r_i \sin^{i-1}\beta \tag{15}$$

and

$$\left[r^2 + \left(\frac{dr}{d\beta}\right)^2\right]^{-\frac{1}{2}} = \frac{1}{a} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} D^k\ , \quad (D \ll 1) \tag{16}$$

wherein

$$D = 2 \sum_{i=0}^{\overline{M}} r_i \sin^i\beta + \left(\sum_{i=0}^{\overline{M}} r_i \sin^i\beta\right)^2$$

$$+ (1 - \sin^2\beta) \left(\sum_{i=1}^{\overline{M}} i\ r_i \sin^{i-1}\beta\right)^2\ , \tag{17}$$

to the equation

$$\sin B = \sin \beta + \sum_{i=0}^{\infty} b_i \sin^i\beta\ . \tag{18}$$

Because the power series in sin β is convergent in the interval

$$-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$$

(it is also absolutely and uniformly convergent therein) the sum of the coefficients b_i is zero or, in detail:

$$\begin{aligned} &\sum_{i=0}^{\infty} b_i = 0 \\ &\sum_{i=0}^{\infty} b_{2i} = 0 \\ &\sum_{i=0}^{\infty} b_{2i+1} = 0 \;. \end{aligned} \tag{19}$$

The last two results can be derived from (18), which is valid for positive and negative values of the geocentric latitude.

As an example we mention again the explicit expressions of the first few coefficients b_i

$$\left.\begin{aligned} b_0 &= - r_1 + \ldots \\ b_1 &= - 2r_2 + \ldots \\ b_2 &= 2r_1 - 3r_3 + \frac{x_1}{2}(2r_2 - 1) + r_1 \frac{x_2}{2} + \ldots \end{aligned}\right\} \tag{20}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

with

$$\left.\begin{aligned} &(x_0 \approx 0) \\ x_1 &= 2r_1 + 4r_1 r_2 + \ldots \\ x_2 &= 2r_2 + 4r_2^2 + \ldots \end{aligned}\right\} \tag{21}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

The geocentric latitude β can be explicitly computed as a function of the geographic latitude B by inversion of equation (18):

$$\sin\beta = \sin B + \sum_{i=0}^{\infty} c_i \sin^i B \,. \tag{22}$$

This series is convergent within the interval (Knopp, 1922)

$$-\frac{\pi}{2} \le B \le \frac{\pi}{2} \,.$$

Similarly to equation (19), we find:

$$\left.\begin{aligned} &\sum_{i=0}^{\infty} c_i = 0 \\ &\sum_{i=0}^{\infty} c_{2i} = 0 \\ &\sum_{i=0}^{\infty} c_{2i+1} = 0, \end{aligned}\right. \tag{23}$$

whereas the first few coefficients c_i read:

$$c_i = c_i(c_0, c_1 \dots c_{i-1}; V) \tag{24}$$

or

$$\left.\begin{aligned} c_0 &= -b_0 + \dots \\ c_1 &= -\frac{b_1}{1+b_1} + \dots \\ c_2 &= -\frac{1}{1+b_1}(b_2 + 2b_2c_1 + 3b_3c_0) + \dots \\ c_3 &= \dots \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{25}$$

If we put B = 0 in equation (22), we obtain the geocentric latitude β_e for the equatorial curve

$$\sin \beta_e = c_0 \quad . \tag{26}$$

This result can be interpreted as a displacement of the gravity center of the equipotential surface of revolution relative to the equatorial plane. The distance in direction of the axis of rotation is therefore:

$$h = a \operatorname{tg} \beta_e = \frac{a\, c_0}{\sqrt{1 - c_0^2}} \quad . \tag{27}$$

3. <u>The potential of the revolution surface, when the equatorial (parallel) radius is known.</u>

All the previous calculations depend in one way or another on the not-yet-known geopotential V. This value can be computed with the help of equations (6) and (27). Putting $\bar{\beta} = \beta_e$ we obtain

$$V = \frac{\mu}{r_e}\left\{1 + \sum_{n=2}^{\bar{N}}\left(\frac{a}{r_e}\right)^n C_{no}\, P_n(\sin \beta_e)\right\} + \frac{\omega^2 r_e^2}{2} \cos^2\beta_e \tag{28}$$

with

$$r_e = \frac{a}{\cos \beta_e} = \frac{a}{\sqrt{1 - c_0^2}} \quad . \tag{29}$$

The only unknown in the above equation is now the geopotential in question. For numerical computations we obtain it with an accuracy up to 10 digits:

$$V = \frac{\mu}{a}\left(1 + \frac{\omega^2 a^3}{2\mu} - \frac{C_{20}}{2} + \frac{3}{8} C_{40} + \ldots\right)^* \quad (\bar{N} = 4) \ . \tag{30}$$

An explicit solution of any desired accuracy can be obtained by inserting (29) into equation (8):

$$a \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i}\left(-c_0^2\right)^i = a\left(1 + \sum_{i=0}^{\bar{M}} r_i\, c_0^i\right) \tag{31}$$

*by numerical iteration we may get any accurcy.

and developing (31) into a power series of $\frac{\mu}{Va}$:

$$1 = \sum_{i=1} v_i \left(\frac{\mu}{Va}\right)^i . \tag{32}$$

An inversion of (32) leads to the value

$$\frac{\mu}{Va} = \sum_{i=0} w_i \tag{33}$$

and hence to the geopotential V :

$$V = \frac{\mu}{a} \frac{1}{\sum\limits_{i=0} w_i} . \tag{34}$$

4. Equation of the meridian curve when the gravity is known.

The procedure is very similar to those discussed in the previous sections. Starting from equation (6) we obtain by partial differentiations the gravity value g in the geocentric latitude β:

$$g = |\text{grad } V| = \left[\left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \beta}\right)^2\right]^{\frac{1}{2}} \tag{35}$$

when

$$\frac{\partial V}{\partial r} = -\frac{\mu}{r^2}\left\{1 + \sum_{n=2}^{\overline{N}} (n+1)\left(\frac{a}{r}\right)^n C_{no} P_n(\sin \beta)\right\} + \omega^2 r \cos^2\beta \tag{36}$$

and

$$\frac{\partial V}{\partial \beta} = \frac{\mu}{r}\left\{\sum_{n=2}^{\overline{N}} \left(\frac{a}{r}\right)^n C_{no} \frac{dP_n(\sin \beta)}{d\beta}\right\} - \frac{\omega^2 r^2}{2} \sin 2\beta . \tag{37}$$

As in section 1 we construct--with the help of equation (35)--an expression

$$r = a\left(1 + \sum_{i=0}^{\bar{M}} r_i \sin^i \beta\right) \tag{38}$$

and continue formally up to equation (34). If $\bar{B}$ is called the geographical latitude in which the gravity g is known, we obtain the corresponding geocentric latitude $\bar{\beta}$ by:

$$\sin \bar{\beta} = \sin \bar{B} + \sum_{i=0}^{\infty} c_i \sin^i \bar{B} , \tag{39}$$

wherein the coefficients c_i are functions of the unknown equatorial radius a. By introducing equation (39) into (38) and the result into equation (35) we can compute with $\beta = \bar{\beta}$ the equatorial radius in question. The other procedures are analogously the same as already described.

5. Transformation of the geocentric equations to the equatorial system.

All the previous equations relate to a coordinate system with its origin O in the gravity center. As a result there usually appears a constant term such as r_0, b_0, c_0, which can be removed by a transformation to the equatorial system:

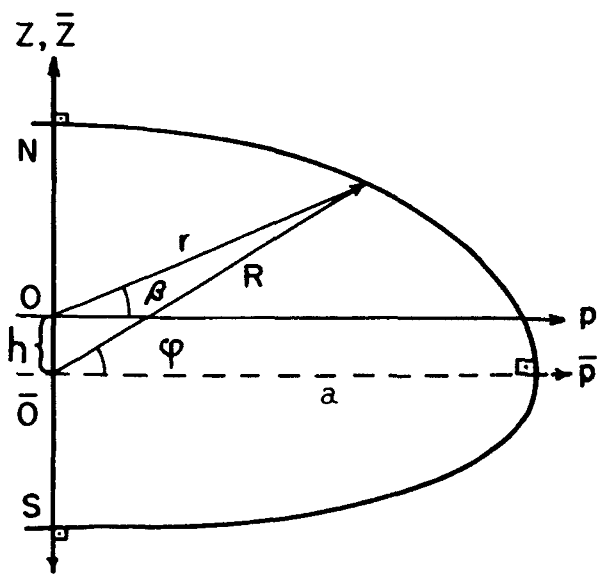

Fig. 2

$$\bar{z} = z + h \tag{40}$$

$$\bar{p} = p \tag{41}$$

with

$$\sin \beta = \frac{z}{r} \; ; \; r = \sqrt{p^2 + z^2} \tag{42}$$

$$\sin \varphi = \frac{\bar{z}}{R} \; ; \; R = \sqrt{\bar{p}^2 + \bar{z}^2} \tag{43}$$

Inserting (40) and (41) in equation (8) we obtain, by observing (42), (43) and

$$\left[\bar{p}^2 + (\bar{z}-h)^2\right]^k = R^{2k} \sum_{j=0}^{\infty} \binom{k}{j}\left(\frac{h}{R} - 2 \sin \varphi\right)^j \left(\frac{h}{R}\right)^j \tag{44}$$

(k any number) ,

an expression similar to equation (8) but without a constant term:

$$R = a\left(1 + \sum_{i=2}^{\infty} R_i \sin^i\varphi\right) . \tag{45}$$

The coefficient R_1 of sin φ in the above equation also becomes zero, which can immediately be seen from equation (14). As an example we mention again the first few expressions of R_i:

$$\left.\begin{aligned} R_2 &= r_2 + \ldots \\ R_3 &= r_3 + 2r_2 \frac{h}{a} + \ldots \\ R_4 &= r_4 + \ldots \end{aligned}\right\} \tag{46}$$

$$\ldots\ldots\ldots\ldots\ldots$$

Corresponding equations can be obtained for (18) and (22) if we substitute R instead of r in the equations under consideration:

$$\sin B = \sin \varphi + \sum_{i=1}^{\infty} \hat{b}_i \sin^i\varphi \tag{47}$$

with

$$\sum_{i=1} \hat{b}_i = 0 \text{, etc.}$$

and

$$\sin \varphi = \sin B + \sum_{i=1}^{\infty} \hat{c}_i \sin^i B \tag{48}$$

with

$$\sum_{i=1} \hat{c}_i = 0 \text{, etc.}$$

However, the difference against the former developments is that R is now an infinite series. Using the following expressions (Ryshik and Gradstein, 1957):

division of power series:

$$\frac{\sum_{i=0}^{\infty} \ell_i x^i}{\sum_{i=0}^{\infty} m_i x^i} = \frac{1}{m_0} \sum_{i=0}^{\infty} n_i x^i \tag{49}$$

with

$$n_j + \frac{1}{m_0} \sum_{i=1}^{j} n_{j-i} m_i - \ell_j = 0 \; ;$$

powers of power series:

$$\left(\sum_{i=0}^{\infty} m_i x^i \right)^k = \sum_{i=0}^{\infty} n_i x^i \quad \text{(k any natural number)} \tag{50}$$

with

$$n_0 = m_0^k ; \; n_j = \frac{1}{j\, m_0} \sum_{i=1}^{j} (ik - j + i)\, m_i\, n_{j-i} \; ;$$

multiplication of power series:

$$\sum_{i=1}^{\infty} \ell_i x^i \sum_{i=1}^{\infty} m_i x^i = \sum_{i=1}^{\infty} n_i x^i \tag{51}$$

with

$$n_j = \sum_{i=1}^{j-1} \ell_i m_{j-i} \; ;$$

we get the coefficients $\hat{b}_i$ and $\hat{c}_i$:

$$\left. \begin{aligned} \hat{b}_1 &= -2R_2 + \ldots \\ \hat{b}_2 &= -3R_3 + \ldots \\ \hat{b}_3 &= \ldots . \end{aligned} \right\} \tag{52}$$

. . . .

and $\quad \hat{c}_i = \hat{c}_i(\hat{c}_1, \hat{c}_2, \ldots \hat{c}_{i-1})$

or

$$\left.\begin{aligned} \hat{c}_1 &= -\frac{\hat{b}_1}{1+\hat{b}_1} + \ldots \\ \hat{c}_2 &= -\frac{\hat{b}_2}{1+\hat{b}_1}(1 + 2\hat{c}_1) + \ldots \\ \hat{c}_3 &= \ldots \\ & \ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{53}$$

In order to obtain the radius R as a function of the geographic latitude B, we insert equation (48) into (45) and, considering the formula for substitution of one power series into another (Ryshik and Gradstein, 1957),

$$\sum_{i=1}^{\infty} m_i\, y^i = \sum_{i=1}^{\infty} n_i\, x^i \quad \text{with} \quad y = \sum_{i=1}^{\infty} \ell_i\, x^i \,, \tag{54}$$

whereby

$$\left.\begin{aligned} n_1 &= \ell_1 m_1 \\ n_2 &= \ell_2 m_1 + \ell_1^2 m_2 \\ n_3 &= \ell_3 m_1 + 2\ell_1 \ell_2 m_2 + \ell_1^3 m_3 \\ & \ldots\ldots\ldots\ldots \end{aligned}\right\} , \tag{55}$$

we get

$$R = a\left(1 + \sum_{i=2}^{\infty} \hat{R}_i \sin^i B\right) \quad \text{with:} \begin{cases} \ell_i \equiv \hat{c}_i \\ m_i \equiv R_i \\ n_i \equiv \hat{R}_i \end{cases} \quad \begin{aligned} y &\equiv \sin\varphi \\ x &\equiv \sin B \,. \end{aligned} \tag{56}$$

According to our former equations the interval of convergence is again:

$$-\frac{\pi}{2} \le B \le \frac{\pi}{2}$$

and

$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \,.$$

6. The parallel radius p as a function of the geographic latitude B.

From figure 2 we learn

$$p^2 = R^2 \cos^2\varphi = a^2 \left(1 + \sum_{i=2}^{\infty} \bar{R}_i \sin^i B\right)^2 (1 - \sin^2\varphi) . \qquad (57)$$

Hence with the help of (48) together with (50) and (51) we obtain the expression in question

$$\left(\frac{p}{a}\right)^2 = 1 + \sum_{i=2}^{\infty} p_i \sin^i B \qquad (58)$$

with

$$\sum_{i=2}^{\infty} p_i = -1; \quad \sum_{i=1}^{\infty} p_{2i} = -1; \quad \sum_{i=1}^{\infty} p_{2i+1} = 0 ,$$

and, for example,

$$\left.\begin{aligned} p_2 &= (2R_2-1)\left[1 + \frac{4R_2}{1-2R_2}\left(1 + \frac{R_2}{1-2R_2}\right)\right] + \ldots \\ p_3 &= 2R_3\left\{1 - \frac{3}{(1-2R_2)^2}\left(8R_2^2 + \frac{1+2R_2}{1-2R_2}\right)\right\} + \ldots \\ p_4 &= \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (59)$$

A slight change of (58) leads to

$$q = \sqrt{1 - \left(\frac{p}{a}\right)^2} = \sin B \left(1 + \sum_{i=0}^{\infty} q_i \sin^i B\right) \qquad (60)$$

with

$$\sum_{i=0}^{\infty} q_i = 0; \quad \sum_{i=0}^{\infty} q_{2i} = 0; \quad \sum_{i=0}^{\infty} q_{2i+1} = 0 .$$

The first few coefficients are

$$\left.\begin{aligned} q_0 &= -\tfrac{1}{2}(1+p_2) - \tfrac{1}{8}(1+p_2)^2 - \tfrac{1}{16}(1+p_2)^3 - \ldots \\ q_1 &= -\tfrac{1}{2}p_3 - \tfrac{1}{4}(1+p_2)p_3 - \ldots \\ q_2 &= \ldots \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (61)$$

By inversion of (60) we obtain the geographic latitude B as a function of q;

$$\sin B = \pm q\left[1 + \sum_{i=0}^{\infty} e_i(\pm q)^i\right] \begin{pmatrix} + \text{ northern hemisphere} \\ - \text{ southern hemisphere} \end{pmatrix} \qquad (62)$$

with

$$\sum_{i=0}^{\infty} e_i = 0; \quad \sum_{i=0}^{\infty} e_{2i} = 0; \quad \sum_{i=0}^{\infty} e_{2i+1} = 0;$$

when

$$\text{or} \quad \left.\begin{aligned} e_i &= e_i(e_0, e_1 \ldots e_{i-1}) \\ e_0 &= -q_0/(1+q_0) + \ldots \\ e_1 &= \frac{q_1}{p_2}/(1+q_0) + \ldots \\ e_2 &= \ldots \\ &\ldots\ldots\ldots \end{aligned}\right\}, \qquad (63)$$

and with (49) we finally get the expression:

$$\frac{1}{\sin B} = \frac{1}{\sqrt{1-\left(\frac{p}{a}\right)^2}}\left[1 + \sum_{i=1}^{\infty} (-1)^i u_{2i}\left(\frac{p}{a}\right)^{2i}\right] \pm \sum_{i=1}^{\infty} (-1)^i v_{2i}\left(\frac{p}{a}\right)^{2i} \qquad (64)$$

$$\begin{pmatrix} + \text{ northern hemisphere} \\ - \text{ southern hemisphere} \end{pmatrix},$$

wherein

$$
\left.
\begin{aligned}
u_2 &= x_2 + 2x_4 + 3x_6 + 4x_8 + \dots \\
u_4 &= x_4 + 3x_6 + 6x_8 + \dots \\
u_6 &= \dots \\
&\dots\dots\dots\dots \\
v_2 &= x_3 + 2x_5 + 3x_7 + \dots \\
v_4 &= x_5 + 3x_7 + \dots \\
v_6 &= \dots \\
&\dots\dots\dots\dots \\
x_2 &= -e_2 + 2e_0e_2 - 3e_0^2e_2 + \dots \\
x_3 &= -e_3 + 2e_0e_3 + 2e_1e_2 + \dots \\
x_4 &= \dots
\end{aligned}
\right\} \qquad (65)
$$

In this form equation (64) will be used for the integration of the differential equations in the next chapter. The interval of convergence of equations (58) and (62) is

$$-\frac{\pi}{2} \leq B \leq \frac{\pi}{2}$$

and

$$0 \leq q \leq 1 \quad .$$

C. The Fundamental Formulas of a Geodesic on an Equipotential Surface of Revolution.

1. Differential equations of the geodesic.

In this section we briefly outline the derivations of the differential equations of geodesics on surfaces of revolution (Köhnlein, 1962).

Starting with the integral formula of Gauss-Bonnet:

$$\iint_{\mathcal{L}} \bar{K}(u,v)\, dF + \oint_{\mathcal{R}} k_g(s)\, ds = 2\pi , \qquad (66)$$

wherein

- u, v = surface parameters
- $\bar{K}$ = Gaussian curvature in a surface point
- dF = surface element
- $\mathcal{L}$ = simply connected region on the surface
- $\mathcal{R}$ = boundary curve of $\mathcal{L}$, steadily curved and without double points
- s = curve length as a parameter of $\mathcal{R}$
- k_g = geodesic curvature of $\mathcal{R}$ as a function of the curve length,,

we obtain for a differential triangle

$$\lim_{\Delta \to 0} \oint_{\Delta} k_g\, ds = 2\pi \quad . \qquad (67)$$

According to figure 3, we introduce on the surface of revolution a coordinate system B,L--with B as the geographic latitude and L as the geographic longitude--thereby calling M the meridian radius and N the radius of the prime vertical. The relation between p and N is given by the

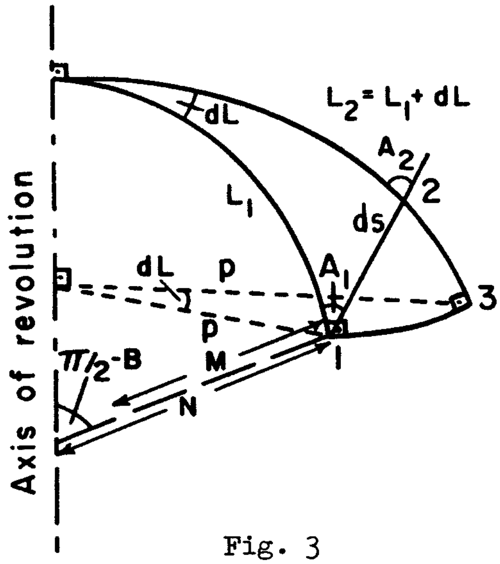

Fig. 3

theorem of Meusnier:

$$p = N \cos B \,. \tag{68}$$

We are considering now a geodesic that includes with the meridian in point 1 the angle A_1 (azimuth). In point 2 we compute the increase of the azimuth angle $dA = A_2 - A_1$ from (67).

$$k_g \; p \; dL + \frac{\pi}{2} + \pi - A_2 + \frac{\pi}{2} + A_1 = 2\pi \tag{69}$$

or

$$k_g \; p \; dL = dA \,. \tag{70}$$

Because the curves ds_{12} and ds_{23} have vanishing geodesic curvature we obtain k_g for the parallel radius with the help of (66):

$$\oint\limits_{p = \text{const.}} k_g \; p \; dL = 2\pi - \int_0^{2\pi} \int_B^{\pi/2} \cos B \; dB \; dL \tag{71}$$

or

$$k_g = \frac{\sin B}{p} \,. \tag{72}$$

Introducing (72) into (70) we get the first differential equation of a geodesic on surfaces of revolution:

$$\frac{dA}{dL} = \sin B \,. \tag{73}$$

The well-known equation of Clairaut can be derived from the triangle 1-2-3 in figure 3.

$$\operatorname{ctg} A = \frac{M \; dB}{p \; dL} \,, \tag{74}$$

and with (73)

$$\frac{dA}{\operatorname{tg} A} = \frac{M \sin B \; dB}{p} = -\frac{dp}{p} \,. \tag{75}$$

Integration of (75)

$$\int_{A_1}^{A_2} \frac{dA}{\operatorname{tg} A} = -\int_{p_1}^{p_2} \frac{dp}{p} \tag{76}$$

finally leads to the expression

$$p_1 \sin A_1 = p_2 \sin A_2 = p_i \sin A_i = p_m \; . \tag{77}$$

The product of the parallel radius p_i and the sine of the azimuth A_i of a geodesic is constant in any point of a revolution surface and equal to the minimal parallel radius p_m $(A_m = \frac{\pi}{2})$.

The second differential equation can be derived again from figure 3 :

$$\sin A = \frac{p\, dL}{ds} \; , \tag{78}$$

or with (73)

$$\frac{ds}{dA} = \frac{p}{\sin A \sin B} \; . \tag{79}$$

2. <u>Integration of the differential equations.</u>

In order to get the difference ΔL of the geographic longitude of a geodesic between two points--with parallel radii p_1 and p_2, respectively--we differentiate (77),

$$dA = -\frac{p_m}{p\sqrt{p^2 + p_m^2}}\, dp \; , \tag{80}$$

and introduce it into (73)

$$\Delta L = L_2 - L_1 = -p_m \int_{p_1}^{p_2} \frac{\sin^{-1} B}{\sqrt{p^2 + p_m^2}} \frac{dp}{p} \; , \tag{81}$$

and similarly we proceed to obtain the length of the geodesic between the said points :

$$s = -\int_{p_1}^{p_2} p \frac{\sin^{-1} B}{\sqrt{p^2 - p_m^2}}\, dp \; . \tag{82}$$

Before we make use of the results of equation (64), we introduce for sake of simplification the equatorial radius a as the unit of length. The formulas with the angle arguments remain unchanged while the distances will be reduced to the said unit.

For equation (81) we now obtain:

$$\Delta L = L_2 - L_1 = - p_m \left\{ \int_{p_1}^{p_2} \frac{dp}{p\sqrt{(1-p^2)(p^2-p_m^2)}} \left[1 + \sum_{i=1}^{\infty} (-1)^i u_{2i} p^{2i} \right] \right.$$
$$\left. \pm \int_{p_1}^{p_2} \frac{dp}{p\sqrt{p^2-p_m^2}} \sum_{i=1}^{\infty} (-1)^i v_{2i} p^{2i} \right\} , \qquad (83)$$

and similarly,

$$s = - \left\{ \int_{p_1}^{p_2} p \frac{dp}{\sqrt{(1-p^2)(p^2-p_m^2)}} \left[1 + \sum_{i=1}^{\infty} (-1)^i u_{2i} p^{2i} \right] \right.$$
$$\left. \pm \int_{p_1}^{p_2} p \frac{dp}{\sqrt{p^2-p_m^2}} \sum_{i=1}^{\infty} (-1)^i v_{2i} p^{2i} \right\} . \qquad (84)$$

The positive sign is valid for the northern hemisphere, and the negative sign is needed for the southern hemisphere.

If we use the abbreviations

$$K = (1-p^2)(p^2-p_m^2) \qquad (85)$$

and

$$k = (p^2 - p_m^2) \quad , \qquad (86)$$

we can compute the above integrals by the recurrence formulas:

$$\int \frac{p^{2t+1}}{\sqrt{K}} dp = - \frac{p^{2(t-1)}}{2t} \sqrt{K} + \frac{2t-1}{2t} \left(1 + p_m^2\right) \int \frac{p^{2t-1}}{\sqrt{K}} dp - \frac{t-1}{t} p_m \int \frac{p^{2t-3}}{\sqrt{K}} dp, \qquad (87)$$

with

$$\int \frac{p}{\sqrt{K}}\, dp = -\tfrac{1}{2} \text{ arc cos } \frac{2p^2 - (1+p_m^2)}{1 - p_m^2}, \tag{88}$$

$$\int \frac{dp}{p\sqrt{K}} = -\frac{1}{2p_m} \text{ arc cos } \frac{(1+p_m^2)\, p^2 - 2p_m^2}{p^2(1-p_m^2)}, \tag{89}$$

and

$$\int \frac{p^{2t+1}}{\sqrt{k}}\, dp = \sum_{i=0}^{t} \binom{t}{i} \frac{k^{t-i+\frac{1}{2}}}{2(t-i)+1}\, p_m^{2i}, \tag{90}$$

with

$$\begin{aligned}
\int \frac{p}{\sqrt{k}}\, dp &= \sqrt{k} \\
\int \frac{p^3}{\sqrt{k}}\, dp &= \frac{1}{3} (\sqrt{k})^3 + p_m^2 \sqrt{k} \\
\int \frac{p^5}{\sqrt{k}}\, dp &= \frac{1}{5} (\sqrt{k})^5 + \frac{2}{3} p_m^2 (\sqrt{k})^3 + p_m^4 \sqrt{k} \\
\int \frac{p^7}{\sqrt{k}}\, dp &= \frac{1}{7} (\sqrt{k})^7 + \frac{3}{5} p_m^2 (\sqrt{k})^5 + p_m^4 (\sqrt{k})^3 + p_m^6 \sqrt{k} \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{91}$$

If the geographic latitude B is for some reason preferred as the variable instead of p, then the above formulas can easily be changed with the help of equation (58).

3. Length of a meridian arc.

The length of the meridian arc between two points with the parallel radii p_1 and p_2 respectively follows from (84) by putting $p_m = 0$. In this special case we obtain:

$$G_{12} = -\left\{ \int_{p_1}^{p_2} \frac{dp}{\sqrt{1-p^2}} \left[1 + \sum_{i=1}^{\infty} (-1)^i \, u_{2i} \, p^{2i} \right] \right.$$

$$\left. \pm \int_{p_1}^{p_2} \left[\sum_{i=1}^{\infty} (-1)^i \, v_{2i} \, p^{2i} \right] dp, \right\} \tag{92}$$

with the recurrence formulas

$$\int \frac{p^{2t}}{\sqrt{1-p^2}} \, dp = - \frac{p^{2t-1}}{2t} \sqrt{1-p^2} + \frac{2t-1}{2t} \int \frac{p^{2n-2}}{\sqrt{1-p^2}} \, dp \ , \tag{93}$$

$$\int \frac{dp}{\sqrt{1-p^2}} = - \text{arc cos } p \ , \tag{94}$$

and

$$\int p^{2t} \, dp = \frac{p^{2t+1}}{2t+1} \ . \tag{95}$$

We notice that G_{12} is also obtained in the length unit of the equatorial radius.

4. Excess of a geodesic polar triangle.

The excess of a geodesic polar triangle is equal to the total curvature of its surface F. It can be computed with the help of (66). Assuming that one corner of the geodesic triangle (polar triangle) coincides with a surface pole, we obtain with equation (83):

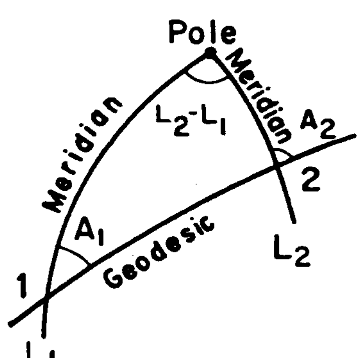

Fig. 4

$$\begin{aligned} \mathcal{E} &= \iint_{\mathcal{L}} \bar{K} \, dF = \iint_{\mathcal{L}} \cos B \, dB \, dL \\ &= + \int_{L_1}^{L_2} (1 - \sin B) \, dL \\ &= - p_m \int_{p_1}^{p_2} \frac{\sin^{-1} B}{\sqrt{p^2 - p_m^2}} \frac{dp}{p} + p_m \int_{p_1}^{p_2} \frac{1}{\sqrt{p^2 - p_m^2}} \frac{dp}{p} \\ &= L_2 - L_1 + A_1 - A_2 \ . \end{aligned} \tag{96}$$

5. Surface area of a geodesic polar triangle.

If $\mathcal{L}$ is again the region within a polar triangle, then we get its area

$$F = \iint_{\mathcal{L}} M\, p\, dB\, dL = - \iint_{\mathcal{L}} \frac{p}{\sin B}\, dp\, dL \,, \tag{97}$$

and with the help of (64)

$$F = -\iint_{\mathcal{L}} \left[\frac{1}{\sqrt{1-p^2}} \sum_{i=0}^{\infty} (-1)^i u_{2i}\, p^{2i+1} \pm \sum_{i=1}^{\infty} (-1)^i v_{2i}\, p^{2i+1} \right] dp\, dL$$

$$= \left[\sum_{i=0}^{\infty} \sum_{j=0}^{i} u_{2i} \binom{i}{j} \frac{(-1)^{2i+j}}{2(i-j)+1} \right] (L_2 - L_1) \tag{98}$$

$$+ \int_{L_1}^{L_2} \left[\sum_{i=0}^{\infty} \sum_{j=0}^{i} u_{2i} \binom{i}{j} \frac{(1-p^2)^{i-j+\frac{1}{2}}}{2(i-j)+1} (-1)^{2i+j+1} \pm \sum_{i=1}^{\infty} (-1)^i v_{2i} \frac{p^{2(i+1)}}{2(i+1)} \right] dL;$$

$$(u_0 \equiv 1) \,.$$

Introducing the expression--see equations (73) and (80)--

$$dL = \frac{dA}{\sin B} = - \frac{p_m}{p\sqrt{p^2-p_m^2}} \frac{1}{\sin B}\, dp \tag{99}$$

$$= - \frac{p_m}{p\sqrt{p^2-p_m^2}} \left[\frac{1}{\sqrt{1-p^2}} \sum_{i=0}^{\infty} (-1)^i u_{2i}\, p^{2i} \pm \sum_{i=1}^{\infty} (-1)^i v_{2i}\, p^{2i} \right] dp \,,$$

$$(u_0 \equiv 1)$$

we finally obtain:

$$F = \left[\sum_{i=0}^{\infty} \sum_{j=0}^{i} u_{2i} \binom{i}{j} \frac{(-1)^{2i+j}}{2(i-j)+1} \right] (L_2 - L_1)$$

$$- p_m \int_{p_1}^{p_2} \frac{dp}{p\sqrt{p^2-p_m^2}} \left[\sum_{i=0}^{\infty} \sum_{j=0}^{i} u_{2i} \binom{i}{j} \frac{(1-p^2)^{i-j+\frac{1}{2}}}{2(i-j)+1} (-1)^{2i+j+1} \right. \tag{100}$$

$$\left. \pm \sum_{i=1}^{\infty} (-1)^i v_{2i} \frac{p^{2(i+1)}}{2(i+1)} \right] \cdot \left[\frac{1}{\sqrt{1-p^2}} \sum_{i=0}^{\infty} (-1)^i u_{2i} p^{2i} \pm \sum_{i=1}^{\infty} (-1)^i v_{2i} p^{2i} \right],$$

wherein the integrals can be computed with the recurrence formulas:

$$\int \frac{p^{2t+1}}{\sqrt{p^2-p_m^2}} = \frac{p^{2t}}{2t+1} \sqrt{p^2-p_m^2} + p_m^2 \frac{2t}{2t+1} \int \frac{p^{2t-1}}{\sqrt{p^2-p_m^2}} dp \tag{101}$$

with

$$\int \frac{dp}{p\sqrt{p^2-p_m^2}} = - \frac{1}{p_m} \text{arc sin} \frac{p_m}{p} \quad . \tag{102}$$

See also formulas (87) to (91).

The area of either the northern or the southern hemisphere of the equipotential surface of revolution can be computed by equating $p_1 = p_2 = p_m = 1$:

$$F_{N,S} = \left[\sum_{i=0}^{\infty} \sum_{j=0}^{i} u_{2i} \binom{i}{j} \frac{(-1)^{2i+j}}{2(i-j)+1} \right.$$

$$\left. \pm \sum_{i=1}^{\infty} (-1)^i \frac{v_{2i}}{2(i+1)} \right] \cdot 2\pi \begin{pmatrix} + \text{ north. hem.} \\ - \text{ south. hem.} \end{pmatrix}, \tag{103}$$

and the area of the whole equipotential surface is therefore

$$F = 4\pi \sum_{i=0}^{\infty} \sum_{j=0}^{i} u_{2i} \binom{i}{j} \frac{(-1)^{2i+j}}{2(i-j)+1} \quad . \tag{104}$$

The results are obtained in the scale unit of the second power of the equatorial radius.

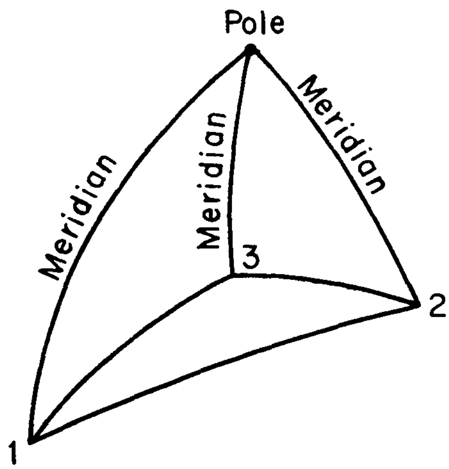

Fig. 5

In sections 4 and 5 we have considered only a geodesic polar triangle. To obtain corresponding results for any geodesic triangle or polygon on the revolution surface, we have only to dissolve the said figures into polar triangles as shown for a triangle in figure 5.

D. <u>Application in Geodesy</u>

1. <u>The direct and inverse geodetic problem</u>.

Most of the geodetic computation procedures in triangulation systems can be reduced to two major problems, the direct geodetic problem and the inverse geodetic problem, which may be interpreted as special cases of coordinate transformations within two geodesic polar coordinate systems (Graf, 1955; Köhnlein, 1962; Köhnlein, 1963).

1.1. <u>The direct geodetic problems</u>:

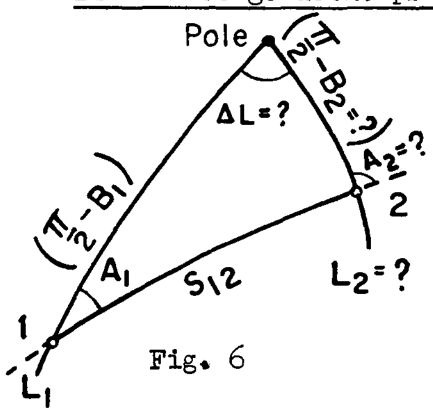

Fig. 6

Assuming that we know the geographic latitude B_1 and longitude L_1 of point 1, then we can compute the coordinates B_2, L_2 of point 2 if the length of the geodesic s_{12} and its azimuth A_1 in point 1 are given.

<u>Solution</u>: The minimal parallel radius p_m can be obtained from equation (77) with

with the help of (58);

$$p_m = p_1 \sin A_1 \; . \tag{105}$$

If we write equation (84) in the shortened form

$$s_{12} = s_{12}\,(p_1, p_2, p_m) \quad , \tag{106}$$

then we find the unknown parallel radius p_2 by developing (106) in a Taylor series of δp_2

$$\delta s_{12} = s_{12} - \hat{s}_{12} = \sum_{i=1} \frac{1}{i!} \, \frac{\partial^i s_{12}}{\partial p_2^i} \, (\delta p_2)^i \; , \tag{107}$$

wherein

δp_2 is the difference of a purely spherically computed parallel radius $\hat{p}_2$ and its actual value p_2;

$$\delta p_2 = p_2 - \hat{p}_2 \; ;$$
$$\left(\hat{p}_2 = \left[1 - (\sqrt{1-p_1^2} \cos s_{12} + p_1 \sin s_{12} \cos A_1)^2 \right]^{\frac{1}{2}} \right) \; ; \tag{108}$$

$\hat{s}_{12}$ is the distance computed with p_1, $\hat{p}_2$, p_m and (84), and

$\dfrac{\partial^i s_{12}}{\partial p_2^i}$ are the partial derivatives also computed with the preliminary value $\hat{p}_2$.

An inversion of equation (107) leads to

$$\delta p_2 = \sum_{i=1} a_i \, \partial s_{12}^i \quad , \tag{109}$$

and hence to

$$p_2 = \hat{p}_2 + \delta p_2 \quad , \tag{110}$$

or, with (62), to B_2, the geographic latitude in question. With the help of (77) we obtain the azimuth A_2,

$$\sin A_2 = \frac{p_m}{p_2} , \tag{111}$$

and finally, with equation (83), the difference ΔL of the geographic longitudes, or

$$L_2 = L_1 + \Delta L = L_1 + \Delta L\,(p_1, p_2, p_m) . \tag{112}$$

If we put $i = 1$ in equation (107), we obtain an iterative solution as described in section 2 (geodesic polygons).

1.2. The inverse geodetic problem:

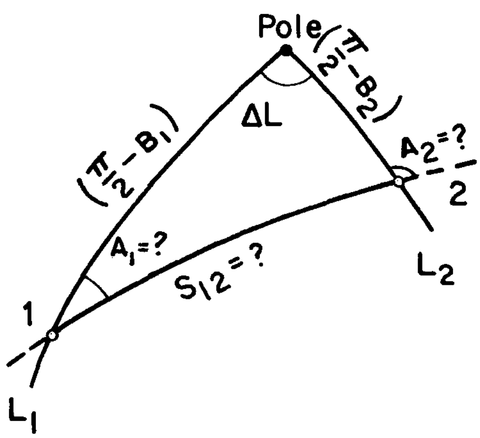

Fig. 7

In this case we know the latitude and longitude B_1, L_1 and B_2, L_2 of points 1 and 2. The problem is to find the azimuth A_1 and A_2 and the distance s_{12} between the two points.

Solution. After computing p_1 and p_2 with the help of (58), we develop, similarly to the previous problem where we used equation (83),

$$\Delta L = L_2 - L_1 = \Delta L(p_1, p_2, p_m) \tag{113}$$

into a Taylor series of δp_m,

$$\delta \Delta L = \Delta L - \hat{\Delta L} = \sum_{i=1} \frac{1}{i!} \frac{\partial^i \Delta L}{\partial p_m^i} (\delta p_m)^i , \tag{114}$$

wherein δp_m is the difference of the purely spherically computed minimal parallel radius $\hat{p}_m$ and its actual value p_m

$$\delta p_m = p_m - \hat{p}_m ,$$

$$\left(\hat{p}_m = \sin \Delta L \Big/ \left[\sin^2 \Delta L + \frac{1-p_1^2}{p_1^2} + \frac{1-p_2^2}{p_2^2} - 2 \frac{\sqrt{(1-p_1^2)(1-p_2^2)}}{p_1 p_2} \cos \Delta L \right]^{\frac{1}{2}} \right) \tag{115}$$

$\widehat{\Delta L}$ is the longitude difference computed with $p_1, p_2, \hat{p}_m$ and (83), and

$\frac{\partial^i \Delta L}{\partial p_m^i}$ are the partial derivatives which are also computed with $\hat{p}_m$.

An inversion of (114) leads to the unknown value δp_m,

$$\delta p_m = \sum_{i=1} b_i \, (\delta \Delta L)^i , \tag{116}$$

and hence to the minimal parallel radius

$$p_m = \hat{p}_m + \delta p_m . \tag{117}$$

With (77) we obtain the azimuths A_1 and A_2,

$$\sin A_1 = \frac{p_m}{p_1} \tag{118}$$

and

$$\sin A_2 = \frac{p_m}{p_2} , \tag{119}$$

while the distance s_{12} can be computed with (84):

$$s_{12} = s_{12} \, (p_1, \, p_2, \, p_m) . \tag{120}$$

An iterative procedure--as applied in high-speed computer technique--is outlined in the next section.

2. Geodesic polygons:

As the direct and inverse geodetic problem has been reduced to a solution of a geodesic polar triangle, we may treat the cases of coordinate transformations in geodesic coordinate systems, etc., by reducing them to the computation of geodesic polygons. For example, a coordinate transformation within two geodesic polar coordinate systems can be solved by means of a geodesic triangle; similarly, the coordinate transformation within two oblique or rectangular geodesic coordinate systems can be solved by geodesic pentagons, etc., etc., (Köhnlein, 1963).

For computing a geodesic polygon we must have 2n-3 angles α_1 and/or sides s_{ii+1}--n is the number of the corners--plus two additional parameters. These two parameters--azimuths or parallel radii etc.,--are necessary to fix the polygon on the surface because of the changing curvature along the meridian curves. Altogether we have 7 n parameters in the polygon: n sides s_{ii+1}, n angles α_i, n parallel radii p_i, n minimal parallel radii $p_{m_{ii+1}}$, n azimuths A_{ii+1}, n azimuths A_{i+1i}, and n values ΔL_{ii+1}. If we like to compute all (excluding the already known values) of them we must have 5n + 1 independent equations, which we write in the shortened form:

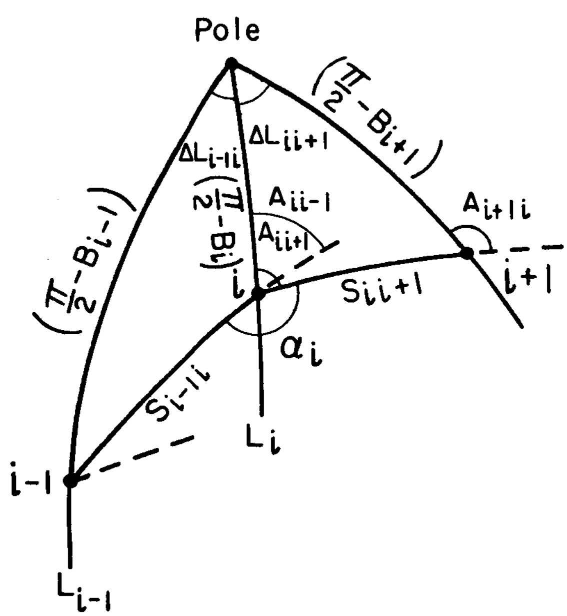

Fig. 8

$$\Delta L_{ii+1} = \Delta L_{ii+1}(p_i, p_{i+1}, p_{m_{ii+1}}) \qquad \text{n equations} \qquad (121)$$

$$s_{ii+1} = s_{ii+1}(p_i, p_{i+1}, p_{m_{ii+1}}) \qquad \text{n equations} \qquad (122)$$

$$p_i \sin A_{ii+1} = p_{i+1} \sin A_{i+1i} = p_{m_{ii+1}} \qquad \text{2n equations} \qquad (123)$$

$$\alpha_i = \pi - (A_{ii+1} - A_{ii-1}) \text{ clockwise!} \qquad \text{n equations,} \qquad (124)$$

wherein $i = 1, 2, \ldots n$ and $n+1 = 1$ (succession of the corners clockwise), and the polygon equation

$$\sum_{i=1}^{n} \Delta L_{ii+1} (p_i, p_{i+1}, p_{m_{ii+1}}) = 0 \qquad \text{1 equation .} \qquad (125)$$

Sometimes it is possible to solve the equations (121)-(125) directly by introducing the known values and computing in steps the values in question. But in most cases we cannot proceed in this way. Writing the above equations in the general form

$$\begin{aligned} f_1(x_1 \ldots x_\nu \ldots x_k) &= 0 \\ \ldots\ldots\ldots\ldots \\ f_\mu(x_1 \ldots x_\nu \ldots x_k) &= 0 \\ \ldots\ldots\ldots\ldots \\ f_k(x_1 \ldots x_\nu \ldots x_k) &= 0 , \end{aligned} \qquad (126)$$

we can reduce the problem to the numerical computation of the values x_ν ($\nu = 1,2 \ldots k$) in (126), which shall be identical with the unknowns in the geodesic polygon. We develop (126) by using Taylor series and the values $\overset{\circ}{x}_\nu = x_\nu - \Delta x_\nu$, and break off after the second term:

$$\begin{aligned} f_\mu(\overset{\circ}{x}_1 + \Delta x_1 \ldots \overset{\circ}{x}_\nu + \Delta x_\nu \ldots \overset{\circ}{x}_k + \Delta x_k) &= f_\mu(\overset{\circ}{x}_1 \ldots \overset{\circ}{x}_\nu \ldots \overset{\circ}{x}_k) \\ + \sum_{\nu=1}^{k} \frac{\partial f_\mu}{\partial x_\nu} (\overset{\circ}{x}_1 \ldots \overset{\circ}{x}_\nu \ldots \overset{\circ}{x}_k) \Delta x_\nu + R_\mu &= 0 \qquad (\mu,\ \nu = 1,2 \ldots k) , \end{aligned} \qquad (127)$$

wherein $\overset{\circ}{x}_\nu$ are approximations of the exact values x_ν, and Δx_ν are small additive corrections. By equating $R_\mu = 0$ we can compute approximate values of $\Delta \overset{1}{x}_\nu$:

$$\left(\Delta \overset{1}{x}_\nu\right)\uparrow = \left\| \bar{a}_{\nu\lambda} \right\| \left(- f_\lambda\right)\uparrow \qquad (\nu, \lambda = 1,2 \ldots k) , \qquad (128)$$

with the square matrix

$$\| a_{\mu\nu} \| = \left\| \begin{matrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_\nu} \cdots \frac{\partial f_1}{\partial x_k} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \frac{\partial f_\mu}{\partial x_1} \cdots \frac{\partial f_\mu}{\partial x_\nu} \cdots \frac{\partial f_\mu}{\partial x_k} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \frac{\partial f_k}{\partial x_1} \cdots \frac{\partial f_k}{\partial x_\nu} \cdots \frac{\partial f_k}{\partial x_k} \end{matrix} \right\| \text{ and } \| a_{\mu\nu} \| \, \| \bar{a}_{\nu\lambda} \| = \| \delta_{\mu\lambda} \| , \qquad (129)$$

wherein

$$\delta_{\mu\lambda} = \begin{cases} 1; & \mu = \lambda \\ 0; & \mu \neq \lambda \end{cases} \quad \text{and the determinant } |a_{\mu\nu}| \neq 0 .$$

Repeating this idea with the improved value

$$\overset{1}{x}_\nu = \overset{0}{x}_\nu + \Delta \overset{1}{x}_\nu \quad (\text{instead of } \overset{0}{x}_\nu) , \qquad (130)$$

we get from (128) new corrections $\Delta \overset{2}{x}_\nu$ which lead after some identical operations to any desired accuracy of the values in question.

From equations (83) and (84) we can easily estimate the influence of the different terms in the final result. Hence it is sufficient for the above computation method to use in (129) only the partial derivatives of the leading first term such as:

$$
\left.
\begin{array}{lll}
\dfrac{\partial \Delta L}{\partial p_m} = \dfrac{1}{1-p_m^2}\sqrt{\dfrac{1-p_i^2}{p_i^2-p_m^2}} & \dfrac{\partial \Delta L}{\partial p_i} = -\dfrac{p_m}{p_i\sqrt{K_i}} & \text{wherein } p_m = p_{i+1} \sin A_{i+1i} \\
 & \dfrac{\partial \Delta L}{\partial p_i} = -\dfrac{p_m \cos A_{ii+1}}{(1-p_m^2)\sqrt{1-p_i^2}} & \quad '' \quad p_m = p_i \sin A_{ii+1} \\
\dfrac{\partial s}{\partial p_m} = \dfrac{p_m^2}{1-p_m^2}\sqrt{\dfrac{1-p_i^2}{p_i^2-p_m^2}} & \dfrac{\partial s}{\partial p_i} = -\dfrac{p_i}{\sqrt{K_i}} & \quad '' \quad p_m = p_{i+1} \sin A_{i+1i} \\
 & \dfrac{\partial s}{\partial p_i} = -\dfrac{\cos A_{ii+1}}{(1-p_m^2)\sqrt{1-p_i^2}} & \quad '' \quad p_m = p_i \sin A_{ii+1} \\
\dfrac{\partial p_m}{\partial A_{ii+1}} = \sqrt{p_i^2-p_m^2} & \dfrac{\partial p_i}{\partial A_{ii+1}} = -\dfrac{p_i}{p_m}\sqrt{p_i^2-p_m^2} & \\
\dfrac{\partial p_{i+1}}{\partial A_{ii+1}} = p_{i+1}\cot A_{ii+1} & K_i = (1-p_i^2)(p_i^2-p_m^2) \quad . &
\end{array}
\right\} \quad (131)
$$

The length unit is again the semimajor axis a.

In order to obtain approximate values $\overset{\circ}{x}_\nu$ for the polygon, we use the spheroidal data in spherical formulas. If the condition for a spherical solution is insufficient, we again obtain the values $\overset{\circ}{x}_\nu$ by changing slightly one or several of the data of the geodesic polygon. For numerical computations it is sufficient to know only good approximations of the coefficients in the matrix $\| a_{\mu\nu} \|$. This is especially important for practical purposes because the matrix has not to be recomputed after each iteration step.

E. Conclusion.

The zonal harmonics of the gravitational field of the Earth have been determined until now from the motion of artificial satellites up to degree nine (twelve) (Izsak, 1963; Kaula, 1963; King-Hele, 1963; Kozai, 1962; etc.). However, the revolution surface derived from these values does not exactly agree with the revolution surface derived from the actual geoid. But the degree of approximation is fairly good, because the oceans, as part of the actual geoid, cover about 70.8 percent of the Earth's surface. A solution of higher accuracy we can get only if we have enough gravity measurements uniformly distributed over the whole world.

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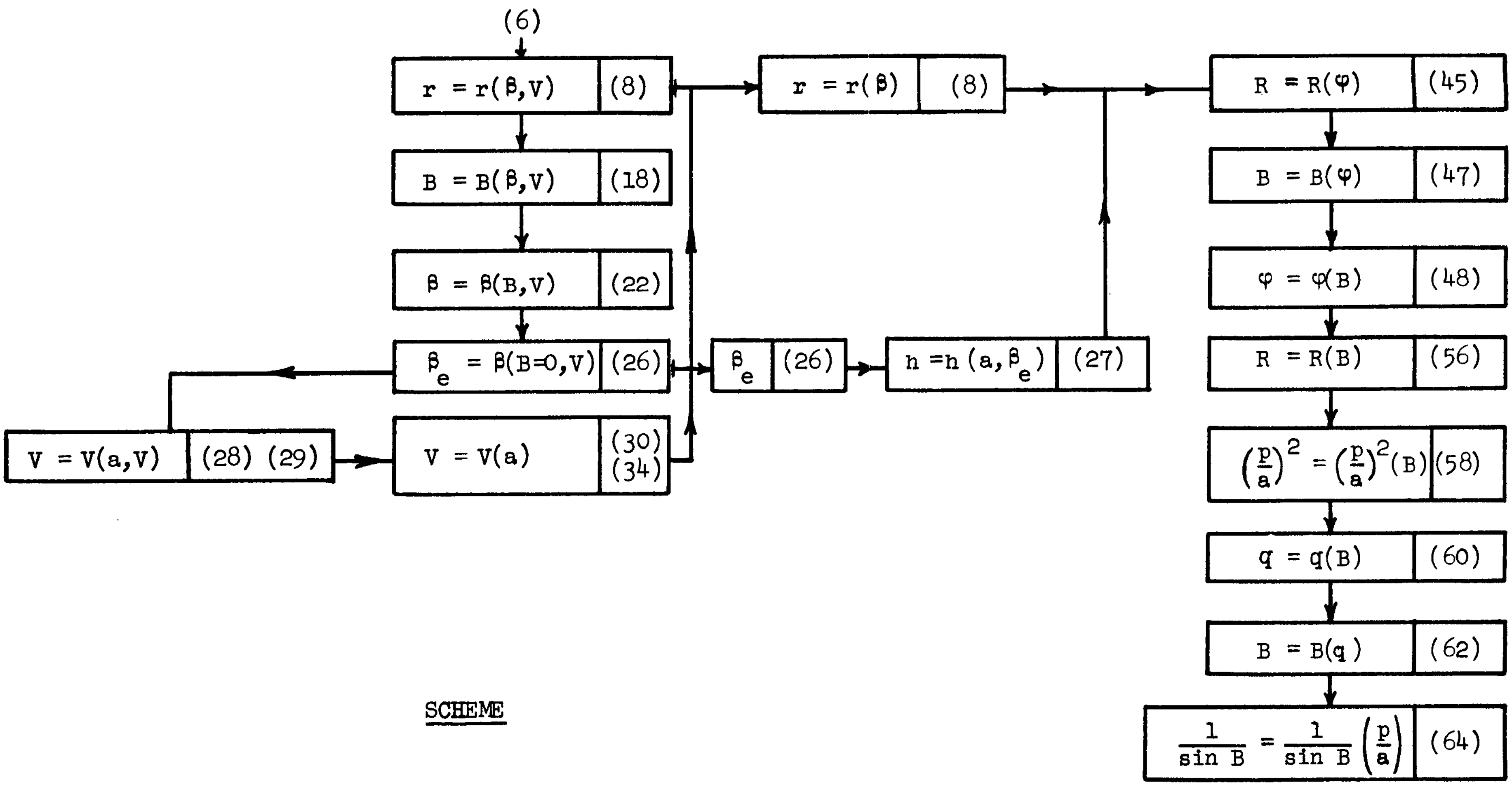
COMPUTATION OF THE MERIDIAN CURVE

(When the equatorial radius a is given)



SCHEME